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(OR MOMENT-STIFFNESS RELATIONS IN CONTINUOUS FRAMES WITH PRISMATIC MEMBERS)

BY

GEORGE HARPER DELL

B.S., Pennsylvania State College, 1922
C.E., Pennsylvania State College, 1926
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PAPERS

MOMENT-STIFFNESS RELATIONS IN CONTINUOUS FRAMES WITH PRISMATIC MEMBERS

BY GEORGE H. DELL,[1] ASSOC. M. ASCE

SYNOPSIS

The design of a continuous frame usually involves a process of successive revisions and analyses, starting with an estimate of the stiffnesses of the various members. On the basis of the stresses obtained from an analysis of this structure, a preliminary design is made. The preliminary design is next analyzed, and it is usually found that some of the members are unnecessarily strong and some are deficient in strength. The designer then makes whatever changes seem appropriate in those members and reanalyzes the structure. This process may have to be repeated several times before the design and the analysis are in agreement.

When the stiffness of one or more members is changed, the moments in all the members are affected. Since the moment at a given section in a continuous structure is a function of the stiffnesses of the various members, the differential of the moment (or moment differential) is a function of the differential changes in the stiffnesses. The resulting relation between the differential of the moment and the differentials of the stiffnesses is designated a "moment-stiffness relation." In some types of structures composed of prismatic members, the analytical part of the design procedure may be facilitated by the use of such moment-stiffness relations. Moment-stiffness relations may also be used in connection with studies aimed at improving the economy of an existing design.

A method of determining the moment-stiffness relations, using moments created by unit couples applied at the several joints, and by unit horizontal forces at the joints in cases involving sidesway, is presented in this paper. The numerical values of the moments resulting from these unit couples and from unit horizontal forces are available for use in obtaining the stresses pro-

NOTE.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by October 1, 1946.

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duced in the original structure by the various combinations of dead load and live load.

The paper also includes a description and illustration of a procedure whereby the moment-stiffness relations can be used to bring the design from the preliminary stage to the final stage.

Since the shears, reactions, and deflections may be obtained from the moments, the differential changes in these quantities may likewise be calculated from the differential changes in the moments, if desired.

The types of structures discussed are those in which the members, assumed to be prismatic, are subjected to bending and shearing forces in a single plane and which may be analyzed from a consideration of the equilibrium of moments and horizontal forces at the various joints—for example, continuous beams, rectangular bents of one or more stories, and saw-tooth bents such as those shown subsequently in Figs. 1(*a*), 2(*a*), 3(*a*), and 5.

Experience indicates that moment-stiffness relations are a useful aid in the design of continuous framed structures composed of prismatic members, particularly in cases where the columns are subject to moments of considerable magnitude. These relations also provide a basis for the determination of possible improvements in an existing design.

As differential quantities are only approximate representations of the true changes, the relative stiffnesses upon which the moment-stiffness relations are based should be chosen with some care, or they should be determined from a preliminary design, and the final results should generally be checked by a direct analysis. It should be recognized that, in some instances, where large stiffness changes occur, the moment differentials will be subject to rather large errors and also that, where only minor changes in stiffness occur, the use of moment-stiffness relations is not particularly advantageous unless subsequent economy studies are to be made.

Conventions and Symbols

The sign convention used for moments is as follows: Tension on the upper side of a horizontal member corresponds to negative moment at the left end and positive moment at the right end; for vertical or inclined members, tension on the left face corresponds to negative moment at the lower end and positive moment at the upper end. Horizontal shears at the ends of vertical or inclined members are considered positive when acting toward the right. Letter symbols are defined where they first appear, and are assembled alphabetically, for convenience of reference, in Appendix 1.

Moment-Stiffness Relations

A general expression for the moment-stiffness relation (that is, the relation between the moment differential and the changes in the stiffnesses of the vari-

ous members) may be derived by analogy with the analytical process used in the determination of the moments. With the general expression as a guide (see Eq. 7), the moment-stiffness relations for a particular structure may be evaluated most conveniently by moment-distribution procedures.

A brief description of the types of structures dealt with herein has been given in the "Synopsis." With regard to any typical member, AB (horizontal, vertical, or inclined), in such structures, the moment at point A is the sum of C_{AB}, the fixed-end moment due to loads between points A and B, and M'_{AB}, the elastic moment or "M'-moment." The latter (M'_{AB}) is equal to the flexural stiffness $\left(K = \frac{E I}{L} \right)$ of the member, times a linear function, ψ, of the elastic deformations. The fixed-end moments being constant, the elastic deformations are functions of the stiffnesses of the various members. The following expression is thus obtained for M_{AB}:

$$M_{AB} = C_{AB} + K_{AB} \psi_{AB} \quad \ldots\ldots (1)$$

The corresponding moment differential is $dM_{AB} = \psi_{AB}\, dK_{AB} + K_{AB}\, d\psi_{AB}$ which, by putting ψ_{AB} equal to M'_{AB}/K_{AB} and dK_{AB} equal to $i_{AB} K_{AB}$, becomes

$$dM_{AB} = M'_{AB} i_{AB} + K_{AB}\, d\psi_{AB} \quad \ldots\ldots (2)$$

In Eq. 2, i is the "stiffness-change ratio."

The horizontal shear at point G in a vertical or inclined member GH, denoted by H_{GH}, is the sum of the simple beam shear and the shear due to the end moments, M_{GH} and M_{HG}. The end moment is the sum of the fixed-end moment C and the M'-moment. Similarly, by defining the fixed-end shear, Q, as the sum of the simple beam shear and the shear created by the fixed-end moments, the horizontal end shear H may be defined as the sum of Q_{GH}, the fixed-end horizontal shear, and H'_{GH}, the horizontal shear due to the M'-moments. Shear H'_{GH} is then equal to the flexural stiffness, K_{GH}, times a linear function, ϕ_{GH}, of the elastic deformations. The following expression is thus obtained for H_{GH}:

$$H_{GH} = Q_{GH} + K_{GH} \phi_{GH} \quad \ldots\ldots (3)$$

The corresponding shear differential is $dH_{GH} = \phi_{GH}\, dK_{GH} + K_{GH}\, d\phi_{GH}$, which, by making ϕ_{GH} equal to H'_{GH}/K_{GH} and dK_{GH} equal to $i_{GH} K_{GH}$, becomes

$$dH_{GH} = H'_{GH} i_{GH} + K_{GH}\, d\phi_{GH} \quad \ldots\ldots (4)$$

An analysis of the moments by either the slope-deflection method or moment-distribution procedures is equivalent to making ΣM equal to zero at the various joints and Σ(horizontal forces) equal to zero for each condition of sidesway or horizontal joint displacement, and solving for the M'-moments, which are then added to the fixed-end moments. Thus, there are two separate classes of equations or operations, which, for convenience, will be designated as "moment equations" and "shear equations." In the case of a moment equation for a given joint J, the constant term, or unbalanced moment, is ΣC_{Jn}—the sum of the fixed-end moments at that joint. In the case of a shear equation

for a given joint or section, the constant term, or unbalanced force, is ΣQ, the sum of the fixed-end horizontal shears, together with any external horizontal forces that may be acting on the joint or section in question.

The moments may be obtained by superposition. Let $m_{A,AB}$, $m_{B,AB}$, . . ., be the moments at joint A in the member AB due to unit couples applied at joints A, B, . . ., respectively; and let $m_{\Delta 1,AB}$, $m_{\Delta 2,AB}$, . . ., be the moments at joint A in the member AB due to unit horizontal shears, or joint forces, corresponding to conditions Nos. 1, 2, . . ., of sidesway or horizontal joint displacement. The following relations may then be stated:

A. The M'-moment at point A in member AB, due to an unbalanced moment of ΣC_{Jn} at joint J, is equal to $m_{J,AB}\,\Sigma C_{Jn}$.

B. The M'-moment at end A in member AB, due to an unbalanced horizontal force of ΣQ_1, is equal to $m_{\Delta 1,AB}\,\Sigma Q_1$.

The total value of M'_{AB} or $K_{AB}\,\psi_{AB}$, in Eq. 1, due to all the unbalanced moments and horizontal forces is thus given by the following expression:

$$M'_{AB} = m_{A,AB}\,\Sigma C_{An} + m_{B,AB}\,\Sigma C_{Bn} + \cdots + m_{\Delta 1,AB}\,\Sigma Q_1 + m_{\Delta 2,AB}\,\Sigma Q_2 + \cdots \quad (5a)$$

The moment M_{AB} may then be expressed as follows:

$$M_{AB} = m_{A,AB}\,\Sigma C_{An} + m_{B,AB}\,\Sigma C_{Bn} + \cdots + m_{\Delta 1,AB}\,\Sigma Q_1 + m_{\Delta 2,AB}\,\Sigma Q_2 + \cdots + C_{AB} \quad (5b)$$

The way is now paved for describing an analogous procedure of obtaining the moment differentials, or moment-stiffness relations. From a comparison of Eqs. 1 and 2, it is noted that the quantity $M'_{AB}\,i_{AB}$, in Eq. 2, corresponds to the fixed-end moment C_{AB}, in Eq. 1; and the quantity $K_{AB}\,d\psi_{AB}$, in Eq. 2, corresponds to the M'-moment, represented by the term $K_{AB}\,\psi_{AB}$, in Eq. 1. Likewise, from a comparison of Eqs. 3 and 4, it is seen that the quantity $H'_{GH}\,i_{GH}$, in Eq. 4, corresponds to the fixed-end shear Q_{GH}, in Eq. 3; and the quantity $K_{GH}\,d\phi_{GH}$, in Eq. 4, corresponds to H'_{GH}, or the term $K_{GH}\,\phi_{GH}$, in Eq. 3.

It can be shown that, just as the moments are obtainable by making ΣM equal to zero at the various joints, and Σ(horizontal forces) equal to zero for each condition of sidesway or horizontal joint displacement, the moment differentials are obtainable by making $\Sigma(dM)$ equal to zero and $\Sigma(dH)$ equal to zero for each moment equation and each shear equation, respectively. Corresponding to the moment equation for a given joint J, the unbalanced moment is $\Sigma(M'_{Jn}\,i_{Jn})$; and, corresponding to the shear equation for a given joint or section, the unbalanced shear is $\Sigma(H'\,i)$.

In connection with the determination of the moment differential dM_{AB} by the principle of superposition, the following relations may now be stated:

A'. The part of the quantity $K_{AB}\,d\psi_{AB}$ in Eq. 2, due to an unbalanced moment of $\Sigma(M'_{Jn}\,i_{Jn})$ at joint J, is equal to $m_{J,AB}\,\Sigma(M'_{Jn}\,i_{Jn})$.

B'. The part of the quantity $K_{AB}\,d\psi_{AB}$ in Eq. 2, due to an unbalanced horizontal shear of $\Sigma(H'_1\,i_1)$, is equal to $m_{\Delta 1,AB}\,\Sigma(H'_1\,i_1)$.

The total value of the quantity $K_{AB}\, d\psi_{AB}$ in Eq. 2, due to all the unbalanced differential moments and horizontal shears, is therefore:

$$K_{AB}\, d\psi_{AB} = m_{A,AB}\, \Sigma(M'_{An}\, i_{An}) + m_{B,AB}\, \Sigma(M'_{Bn}\, i_{Bn}) + \cdots + m_{\Delta 1,AB}\, \Sigma(H'_1\, i_1) + m_{\Delta 2,AB}\, \Sigma(H'_2\, i_2) + \cdots \quad (6)$$

In accordance with Eq.2, the total moment differential, or moment-stiffness relation, for M_{AB} is then as follows:

$$dM_{AB} = m_{A,AB}\, \Sigma(M'_{An}\, i_{An}) + m_{B,AB}\, \Sigma(M'_{Bn}\, i_{Bn}) + \cdots + m_{\Delta 1,AB}\, \Sigma(H'_1\, i_1) + m_{\Delta 2,AB}\, \Sigma(H'_2\, i_2) + \cdots + M'_{AB}\, i_{AB} \quad (7)$$

As a check, it should be noted that the sum of the coefficients of i in a complete moment-stiffness relation is theoretically equal to zero.

Referring to Eq. 7, the procedure for obtaining the moment-stiffness relation for a given monent M_{AB}, due to a particular arrangement of loading producing given M'-moments in the various members, will be clarified by the following rules:

1. Place a unit couple successively at joints A, B, . . ., N and distribute moments throughout the structure in each instance. This step provides the first class of m-moments for the ends of all members in the structure. In obtaining the moment-stiffness relation for M_{AB}, the particular values to be taken from this step are $m_{A,AB}$, $m_{B,AB}$, . . ., $m_{N,AB}$. If the structure is subject to sidesway, the correction for sidesway should be made in finding these m-moments.

Line		A	(2)	B	(4)	C	(3)	D
1	Relative Stiffness, K							
2	Span Length, L		30'		15'		20'	
3	Fixed-End Moments	−400		+400	−100	+100	0	0
4	M'-Moments	+400		+ 56	−356	−133	+33	0
5	Total Moments	0		+456	−456	− 33	+33	0

(*a*)

6	Moments m_A	−1.00	−0.34	+0.34	+0.07	−0.07	0
7	Moments m_B	0	−0.30	−0.70	−0.15	+0.15	0
8	Moments m_C	0	+0.10	−0.10	−0.60	−0.40	0
9	Moments m_D	0	−0.05	+0.05	+0.30	−0.30	−1.00

(*b*)

FIG. 1.—SIMPLY-SUPPORTED CONTINUOUS BEAM

2. Place a unit horizontal force successively at appropriate joints of the structure, corresponding to the various conditions of sidesway or horizontal joint displacement to which the structure is subject, and find the moments thus produced throughout the structure in each instance, when the structure is in equilibrium. This step provides the second class of m-moments for the ends of all members in the structure. For obtaining the moment-stiffness relation for M_{AB}, the particular values to be taken from this step are $m_{\Delta 1,AB}$, $m_{\Delta 2,AB}$, etc.

3. At joint A take the product of $m_{A,AB}$ (see step 1) times the sum of the quantities $M'\, i$ for the various members connecting at that joint; similarly, take products for each joint of the structure that is required to be balanced.

4. Corresponding to each condition of sidesway or horizontal joint displacement to which the structure is subject, take the product of $m_{\Delta, AB}$ (see step 2) times the sum of the quantities H' i for the various members entering into the given sidesway condition.

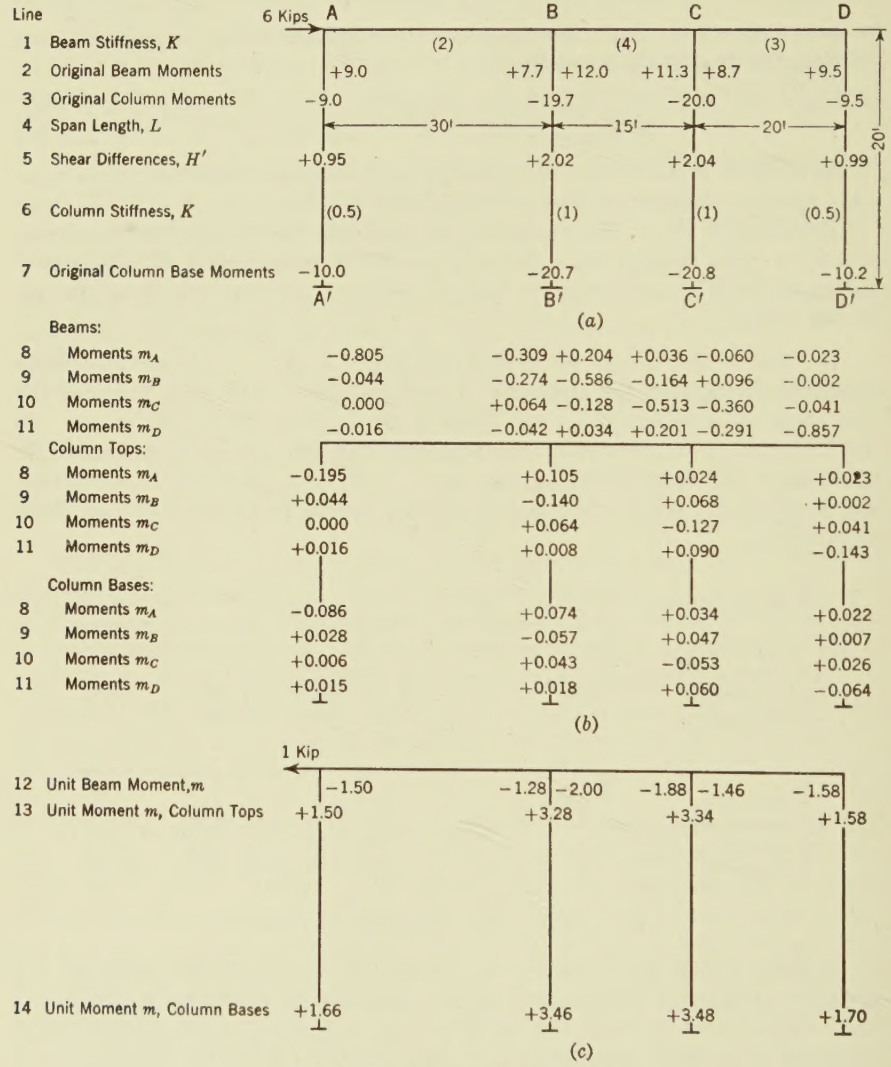
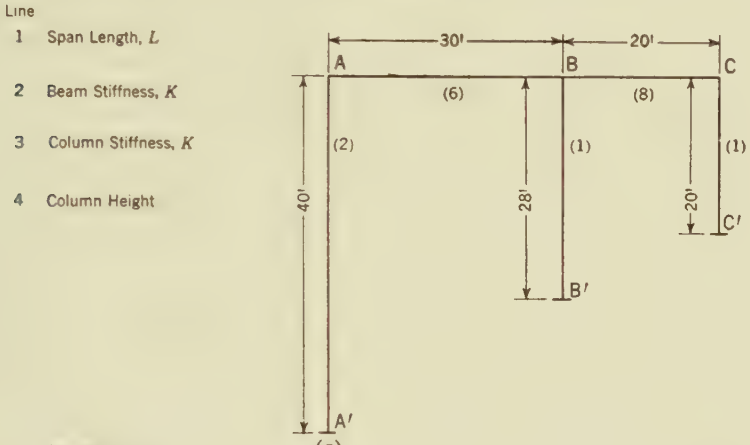


FIG. 2.—RECTANGULAR RIGID FRAME

5. To the sum of all the terms in steps 3 and 4 add the term $M'_{AB}\, i_{AB}$, and place the resulting expression equal to dM_{AB}. A check is obtained by the sum of the coefficients of i, which is theoretically equal to zero.

The application of the foregoing steps to a variety of structures is explained in the following examples, after which the use of moment-stiffness relations in connection with design will be explained and illustrated.

Example 1. Simply-Supported Continuous Beam.—In this problem it is required to calculate the moment-stiffness relation for M_{BA} in Fig. 1(*a*) if the spans AB and BC carry a uniformly distributed load of $5\frac{1}{3}$ kips per ft. Fig. 1(*a*) shows the fixed-end moments, the M'-moments, and the total moments. The relative stiffnesses are shown in parentheses (line 1, Fig. 1(*a*)). The m-moments in Fig. 1(*b*) were obtained by placing a unit counterclockwise couple at successive joints. In accordance with Eq. 7, the required moment-stiffness relation is: $dM_{BA} = -0.34\ (400\ i_{AB}) - 0.30\ (56\ i_{AB} - 356\ i_{BC}) + 0.10\ (-133\ i_{BC} + 33\ i_{CD}) - 0.05\ (0\ i_{CD}) + 56\ i_{AB}$; or $dM_{BA} = -96.8\ i_{AB} + 93.5\ i_{BC} + 3.3\ i_{CD}$. As a check: $-96.8 + 93.5 + 3.3 = 0$.



Line					
	Beams:				
5	Moments m_A	−0.767	−0.242	+0.174	−0.033
6	Moments m_B	−0.060	−0.403	−0.514	−0.034
7	Moments m_C	−0.007	+0.163	−0.232	−0.912
8	Moments m_Δ	−4.070	−1.820	−2.110	−4.820
	Column Tops:				
5	Moments m_A	−0.233	+0.068		+0.033
6	Moments m_B	+0.060	−0.084		+0.034
7	Moments m_C	+0.007	+0.069		−0.088
8	Moments m_Δ	+4.070	+3.930		+4.820
	Column Bases:				
5	Moments m_A	−0.090	+0.053		+0.043
6	Moments m_B	+0.028	−0.043		+0.015
7	Moments m_C	+0.026	+0.050		−0.021
8	Moments m_Δ	+4.760	+3.920		+5.140

(*b*)

FIG. 3.—RIGID FRAME WITH UNEQUAL LEGS

Example 2. Rectangular Rigid Frame.—This problem requires the computation of the moment-stiffness relations for M_{BC} and $M_{A'A}$ for the loading arrangement in Fig. 2(*a*). The moments produced in the original structure by the horizontal force of 6 kips are shown in Fig. 2(*a*). All the moments in

Fig. 2(*a*) are also M'-moments. The fixed-end moments C are all zero, since there are no intermediate loads. The values of H' at the tops of the columns are written in line 5, Fig. 2(*a*).

The m-moments due to a unit couple at joints A, B, C, and D, respectively, are shown in Fig. 2(*b*). These moments have been corrected for sidesway. Fig. 2(*c*) gives the m-moments due to a unit horizontal force at the top of column AA′.

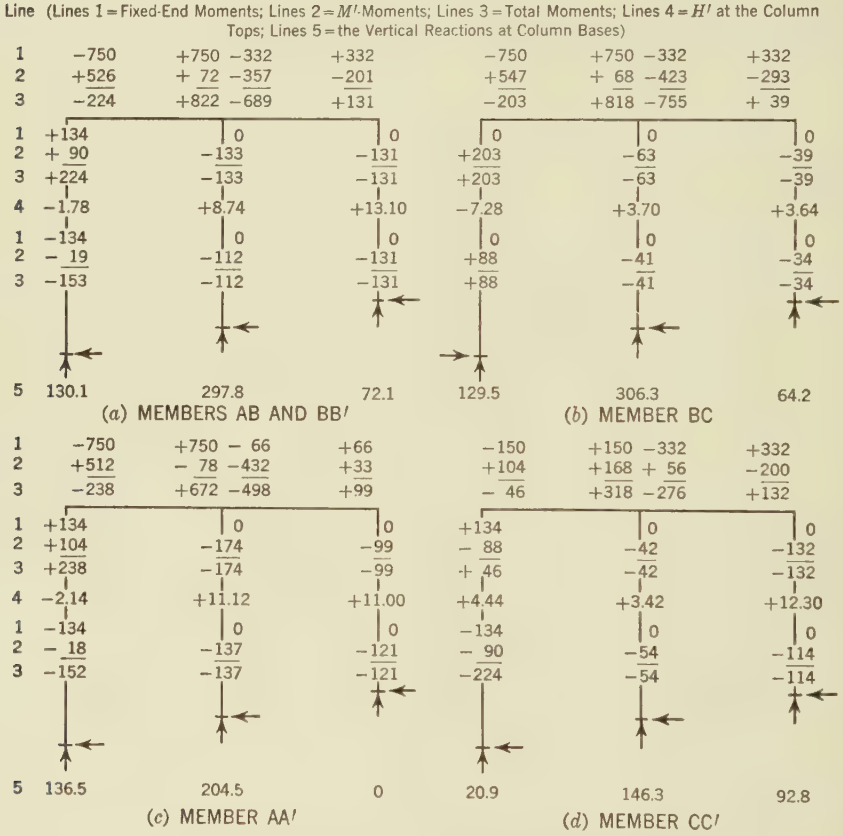

Line (Lines 1 = Fixed-End Moments; Lines 2 = M′-Moments; Lines 3 = Total Moments; Lines 4 = H′ at the Column Tops; Lines 5 = the Vertical Reactions at Column Bases)
(a) MEMBERS AB AND BB′
(b) MEMBER BC
(c) MEMBER AA′
(d) MEMBER CC′

Fig. 4.—Controlling Arrangement for the Analysis of Members in Fig. 3

In the following calculation of the required moment-stiffness relations (which is made in accordance with Eq. 7), the members AA′, AB, BB′, BC, CC′, CD, and DD′ are denoted by the subscripts 1 to 7, respectively: $dM_{BC} = +0.204\ (-9.0\,i_1 + 9.0\,i_2) - 0.586\ (7.7\,i_2 - 19.7\,i_3 + 12.0\,i_4) - 0.128\ (11.3\,i_4 - 20.0\,i_5 + 8.7\,i_6) + 0.034\ (9.5\,i_6 - 9.5\,i_7) - 2.00\ (0.95\,i_1 + 2.02\,i_3 + 2.04\,i_5 + 0.99\,i_7) + 12.0\,i_4$. Simplifying:

$$dM_{BC} = -3.7\,i_1 - 2.7\,i_2 + 7.5\,i_3 + 3.6\,i_4 - 1.5\,i_5 - 0.8\,i_6 - 2.3\,i_7 \ldots (8a)$$

As a check: $-3.7 - 2.7 + 7.5 + 3.6 - 1.5 - 0.8 - 2.3 = +0.1$. Similarly:

$$dM_{A'A} = -0.086\,(-9.0\,i_1 + 9.0\,i_2) + 0.028\,(7.7\,i_2 - 19.7\,i_3 + 12.0\,i_4) + 0.006\,(11.3\,i_4 - 20.0\,i_5 + 8.7\,i_6) + 0.015\,(9.5\,i_6 - 9.5\,i_7) + 1.66\,(0.95\,i_1 + 2.02\,i_3 + 2.04\,i_5 + 0.99\,i_7) - 10.0\,i_1.$$

Simplifying:

$$dM_{A'A} = -7.6\,i_1 - 0.6\,i_2 + 2.7\,i_3 + 0.4\,i_4 + 3.3\,i_5 + 0.2\,i_6 + 1.5\,i_7 \quad (8b)$$

As a check to Eq. 8*b*: $-7.6 - 0.6 + 2.7 + 0.4 + 3.3 + 0.2 + 1.5 = -0.1$.

Example 3. Rigid Frame with Unequal Legs.—In Fig. 3(*a*), showing a bent with three legs of unequal length, it is required to calculate the moment-stiffness relation for M_{BA} when both spans carry a uniform dead load of 2 kips per ft and a uniform live load of 8 kips per ft, and the column AA′ (Fig. 3(*a*)) is subjected to uniform live load of 1 kip per ft.

Fig. 4(*a*) shows the fixed-end moments (lines 1), the M'-moments (lines 2), and the total moments (lines 3) and also the values of H' at the tops of the columns (lines 4). The m-moments, m_A, m_B, m_C, and m_Δ, are given in Fig. 3(*b*).

In the following calculation of the required moment-stiffness relation, by Eq. 7, the members AA′, AB, BB′, BC, and CC′ are denoted by the subscripts 1 to 5, respectively: $dM_{BA} = -0.242\,(90\,i_1 + 526\,i_2) - 0.403\,(72\,i_2 - 133\,i_3 - 357\,i_4) + 0.163\,(-201\,i_4 - 131\,i_5) - 1.82\,(-1.78\,i_1 + 8.74\,i_3 + 13.10\,i_5) + 72\,i_2$; which, simplified, becomes

$$dM_{BA} = -18.6\,i_1 - 84.4\,i_2 + 37.4\,i_3 + 110.8\,i_4 - 45.2\,i_5 \quad (9)$$

As a check: $-18.6 - 84.4 + 37.4 + 110.8 - 45.2 = 0$.

The moment-stiffness relation in Eq. 9 is shown in tabular form in the first line of Table 1.

Example 4. Two-Span, Saw-Tooth Bent.—Calculate the moment-stiffness relations for M_{AB} and $M_{CC'}$, in Fig. 5(*a*), when the spans carry a uniformly distributed load of 1 kip per ft of horizontal projection. The analyses shown in Figs. 5 were obtained by alternate distributions of moments and shears. The procedure is briefly described in Appendix 2. Fig. 5(*a*) shows the fixed-end moments (lines 2), the M'-moments (lines 3), and the total moments (lines 4). The m-moments due to unit couples at joints A, B, C, D, and E are shown in Fig. 5(*b*), lines 5, 6, 7, 8, and 9, respectively. Fig. 5(*c*) gives the m-moments due to unit horizontal forces at points A, C, and E, indicated by $m_{\Delta A}$, $m_{\Delta C}$, and $m_{\Delta E}$ in lines 10, 11, and 12, respectively.

TABLE 1.—Moment-Stiffness Relations

Moment	i_1	i_2	i_3	i_4	i_5
M_{BA}	−18.6	− 84.4	+37.4	+110.8	−45.2
M_{BC}	+50.8	+ 60.2	+24.6	−137.6	+ 1.4
$M_{AA'}$	+71.0	−123.8	+34.8	− 25.8	+44.2
$M_{BB'}$	− 0.8	+ 29.8	−87.4	+ 16.2	+42.4
$M_{CC'}$	+18.4	+ 9.0	+15.0	+ 19.4	−61.2

The members AA′, AB, BC, CC′, CD, DE, and EE′ will be denoted by the subscripts 1 to 7, respectively. In the application of Eq. 7 to the structure of Fig. 5, the terms indicated by $\Sigma(H'_1\,i_1)$ $\Sigma(H'_2\,i_2)$, etc., require particular attention. The horizontal end shears at joints A, C, and E must be kept in

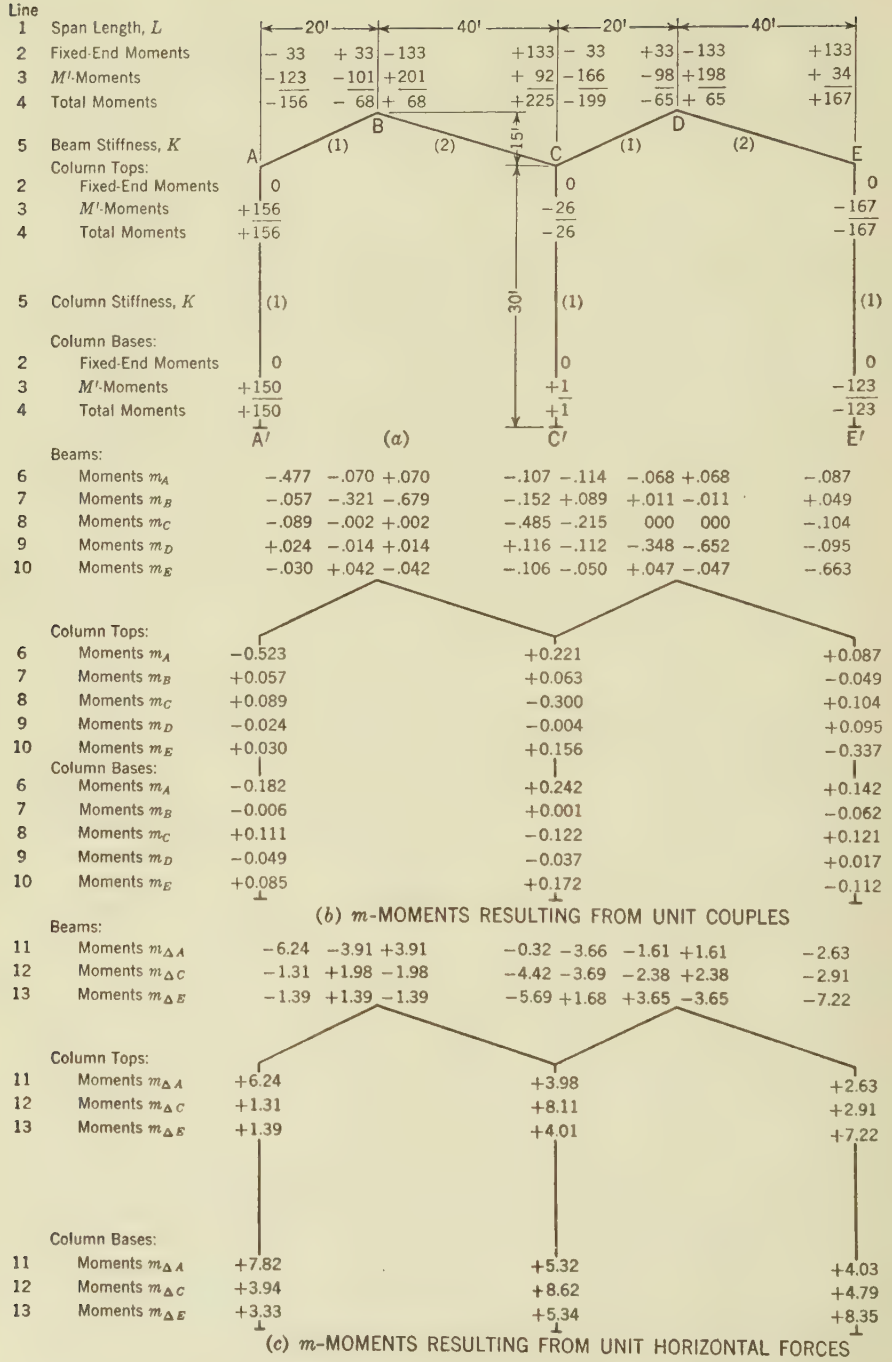

Line
1 Span Length, L
20'
40'
20'
40'
2 Fixed-End Moments
− 33 + 33 −133 +133 − 33 +33 −133 +133
3 M'-Moments
−123 −101 +201 + 92 −166 −98 +198 + 34
4 Total Moments
−156 − 68 + 68 +225 −199 −65 + 65 +167
5 Beam Stiffness, K
A
B
C
D
E
(1)
(2)
(1)
(2)
15'
Column Tops:
2 Fixed-End Moments
0 0 0
3 M'-Moments
+156 −26 −167
4 Total Moments
+156 −26 −167
5 Column Stiffness, K
(1) (1) (1)
30'
Column Bases:
2 Fixed-End Moments
0 0 0
3 M'-Moments
+150 +1 −123
4 Total Moments
+150 +1 −123
A'
C'
E'
(a)
Beams:
6 Moments m_A −.477 −.070 +.070 −.107 −.114 −.068 +.068 −.087
7 Moments m_B −.057 −.321 −.679 −.152 +.089 +.011 −.011 +.049
8 Moments m_C −.089 −.002 +.002 −.485 −.215 000 000 −.104
9 Moments m_D +.024 −.014 +.014 +.116 −.112 −.348 −.652 −.095
10 Moments m_E −.030 +.042 −.042 −.106 −.050 +.047 −.047 −.663
Column Tops:
6 Moments m_A −0.523 +0.221 +0.087
7 Moments m_B +0.057 +0.063 −0.049
8 Moments m_C +0.089 −0.300 +0.104
9 Moments m_D −0.024 −0.004 +0.095
10 Moments m_E +0.030 +0.156 −0.337
Column Bases:
6 Moments m_A −0.182 +0.242 +0.142
7 Moments m_B −0.006 +0.001 −0.062
8 Moments m_C +0.111 −0.122 +0.121
9 Moments m_D −0.049 −0.037 +0.017
10 Moments m_E +0.085 +0.172 −0.112
(b) m-MOMENTS RESULTING FROM UNIT COUPLES
Beams:
11 Moments $m_{\Delta A}$ −6.24 −3.91 +3.91 −0.32 −3.66 −1.61 +1.61 −2.63
12 Moments $m_{\Delta C}$ −1.31 +1.98 −1.98 −4.42 −3.69 −2.38 +2.38 −2.91
13 Moments $m_{\Delta E}$ −1.39 +1.39 −1.39 −5.69 +1.68 +3.65 −3.65 −7.22
Column Tops:
11 Moments $m_{\Delta A}$ +6.24 +3.98 +2.63
12 Moments $m_{\Delta C}$ +1.31 +8.11 +2.91
13 Moments $m_{\Delta E}$ +1.39 +4.01 +7.22
Column Bases:
11 Moments $m_{\Delta A}$ +7.82 +5.32 +4.03
12 Moments $m_{\Delta C}$ +3.94 +8.62 +4.79
13 Moments $m_{\Delta E}$ +3.33 +5.34 +8.35
(c) m-MOMENTS RESULTING FROM UNIT HORIZONTAL FORCES

FIG. 5.—TWO-SPAN, RIGID FRAME WITH A UNIFORMLY DISTRIBUTED LOAD OF 1 KIP PER FOOT ON SPANS

equilibrium: hence, at point A—

$$\Sigma(H'\,i) = -\frac{M'_{AA'} + M'_{A'A}}{h_1}\,i_1 + \frac{M'_{AB} + M'_{BA}}{h'_2}\,i_2 - \frac{M'_{BC} + M'_{CB}}{h'_3}\,i_3 \quad (10a)$$

at point C—

$$\Sigma(H'\,i) = -\frac{M'_{AB} + M'_{BA}}{h'_2}\,i_2 + \frac{M'_{BC} + M'_{CB}}{h'_3}\,i_3 - \frac{M'_{CC'} + M'_{C'C}}{h_4}\,i_4 + \frac{M'_{CD} + M'_{DC}}{h'_5}\,i_5 - \frac{M'_{DE} + M'_{ED}}{h'_6}\,i_6 \quad (10b)$$

and, at point E—

$$\Sigma(H'\,i) = -\frac{M'_{CD} + M'_{DC}}{h'_5}\,i_5 + \frac{M'_{DE} + M'_{ED}}{h'_6}\,i_6 - \frac{M'_{EE'} + M'_{E'E}}{h_7}\,i_7 \quad (10c)$$

In Eqs. 10, h_1, h_4, and h_7 are the heights of the columns AA', CC', and EE', respectively. The meanings of h'_2 and h'_3 are illustrated in Fig. 6; h'_5 and h'_6 in the right-hand span are analogous to h'_2 and h'_3 in the left-hand span, respectively. In Fig. 5, $h'_2 = h'_5 = 22.5$ ft, and $h'_3 = h'_6 = 45$ ft. The required moment-stiffness relations are then as follows: $dM_{AB} = -0.477\,(156\,i_1 - 123\,i_2) - 0.057\,(-101\,i_2 + 201\,i_3) - 0.089\,(92\,i_3 - 26\,i_4 - 166\,i_5) + 0.024\,(-98\,i_5 + 198\,i_6) - 0.030\,(34\,i_6 - 167\,i_7) - 6.24\,(-10.2\,i_1 - 10.0\,i_2 - 6.50\,i_3) - 1.31\,(10.0\,i_2 + 6.50\,i_3 + 0.83\,i_4 - 11.7\,i_5 - 5.17\,i_6) - 1.39\,(11.7\,i_5 + 5.17\,i_6 + 9.67\,i_7) - 123\,i_2$. Simplifying:

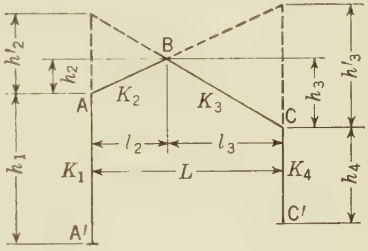


Fig. 6

$$dM_{AB} = -10.8\,i_1 - 9.3\,i_2 + 12.5\,i_3 + 1.2\,i_4 + 11.6\,i_5 + 3.3\,i_6 - 8.4\,i_7 \quad (11a)$$

As a check: $-10.8 - 9.3 + 12.5 + 1.2 + 11.6 + 3.3 - 8.4 = +0.1$. Also, $dM_{CC'} = +0.221\,(156\,i_1 - 123\,i_2) + 0.063\,(-101\,i_2 + 201\,i_3) - 0.300\,(92\,i_3 - 26\,i_4 - 166\,i_5) - 0.004\,(-98\,i_5 + 198\,i_6) + 0.156\,(34\,i_6 - 167\,i_7) + 3.98\,(-10.2\,i_1 - 10.0\,i_2 - 6.50\,i_3) + 8.11\,(10.0\,i_2 + 6.50\,i_3 + 0.83\,i_4 - 11.7\,i_5 - 5.17\,i_6) + 4.01\,(11.7\,i_5 + 5.17\,i_6 + 9.67\,i_7) - 26\,i_4$. Again simplifying:

$$dM_{CC'} = -6.1\,i_1 + 7.7\,i_2 + 11.9\,i_3 - 11.5\,i_4 + 2.1\,i_5 - 16.8\,i_6 + 12.7\,i_7 \quad (11b)$$

As a check: $-6.1 + 7.7 + 11.9 - 11.5 + 2.1 - 16.8 + 12.7 = 0$.

Application to Design

The following outline, although subject to variation, should serve as a general guide in designing with the aid of moment-stiffness relations:

(1) The relative stiffnesses upon which the moment-stiffness relations are to be based are chosen carefully, or they are determined from a preliminary design. The structure thus obtained is called the "basic structure."

(2) The basic structure is analyzed to determine (*a*) the *m*-moments due to unit couples placed at successive joints; and (*b*) the *m*-moments due to unit

horizontal forces or shears corresponding to the various sidesway conditions existing in the structure. If sidesway conditions are present, correction for sidesway should be made in finding the former group of m-moments.

(3) Analyses of the basic structure are made to determine (*a*) the loading arrangements controlling the design of the various members; (*b*) the values of the corresponding controlling moments and axial forces; and (*c*) the M'-moments and values of H' needed for the evaluation of the moment-stiffness relations. The performance of these analyses is expedited by the use of the m-moments obtained in step (2).

(4) The members are designed for the controlling stresses obtained in step (3). The resulting stiffnesses of the members will be different, in general, from those of the basic structure. The stiffness-change ratio, i, or the change in stiffness divided by the stiffness assumed in the basic structure, is calculated for each member.

(5) Moment-stiffness relations corresponding to the various controlling moments are calculated, and are tabulated in such a way that the coefficients of i for a given member all appear in the same column, as shown in Table 1. (The changes in the axial forces in vertical or inclined compression members, due to subsequent changes in the stiffnesses of the various members, generally do not appreciably affect the design and therefore may usually be disregarded.)

(6) By reference to the tabulated moment-stiffness relations, the changes in all the controlling moments, due to a change of stiffness in a given member (from step (4)), are computed by a single setting of the slide rule. Similarly, the moment changes due to the changes in stiffness of the various other members are calculated. In the performance of this step the designer is rewarded for the labor of obtaining the moment-stiffness relations. With their aid, he is able not only to calculate, rapidly, the effects of any number of consecutive revisions in the design, but to form an advance estimate of the effects of contemplated revisions.

(7) The members are redesigned in accordance with the revised moments obtained in step (6), and the moment changes thus produced are again computed, as in step (6). This process is continued until no further changes in design are required. The stiffness-change ratios are always based upon the stiffnesses assumed in the basic structure.

Example 5.—The bent outlined in Fig. 3(*a*) is to be designed in steel in accordance with the 1936 specifications of the American Institute of Steel Construction (AISC) revised in 1941. The horizontal members carry a dead load of 2 kips per ft and a live load of 8 kips per ft. Provision is also to be made for a live load of 1 kip per ft applied uniformly to the side of column AA′. The weight of the columns will be neglected. The unsupported lengths of the columns are: AA′, 13½ ft; BB′, 14 ft; and CC′, 20 ft.

A description of the design procedure conforming to steps (1) to (7) is as follows:

(1) The basic structure and assumed stiffnesses are as shown in Fig. 3(*a*).

(2) The m-moments, m_A, m_B, m_C, and m_Δ are shown in Fig. 3(*b*).

(3*a*) Figs. 4 contain the analyses corresponding to the loading arrangements that were found to control the design of the members. In Fig. 4(*a*), the loading, which controls the design of members AB and BB′, consists of dead load plus live load on both spans, and lateral live load. In Fig. 4(*b*), the loading that controls the design of member BC consists of dead load plus live load on both spans, but no lateral live load. The loading in Fig. 4(*c*) consists of dead load plus live load on the left span, dead load on the right span, and lateral live load. This loading controls the design of member AA′. The loading in Fig. 4(*d*) consists of dead load on the left span, dead load plus live load on the right span, and lateral live load. This loading controls the design of member CC′.

(3*b*) and (3*c*) Figs. 4 show the fixed-end moments, the M'-moments, and the total moments, in the order named. These analyses were made with the aid of the m-moments shown in Fig. 3(*b*), utilizing relations A and B which were stated in connection with the development of Eq. 5*b*. The values of H' at the tops of the columns are entered in lines 4 in Figs. 4.

The controlling moments (M) and axial forces (F) are as follows: For member AB, $M = 822$ ft-kips (line 3); for member BB′, $M = 133$ ft-kips (line 3) and $F = 297.8$ kips (line 5), as shown in Fig. 4(*a*); for member BC, $M = 755$ ft-kips (line 3), as shown in Fig. 4(*b*); for member AA′, $M = 238$ ft-kips (line 3) and $F = 136.5$ kips (line 5), as shown in Fig. 4(*c*); and for member CC′, $M = 132$ ft-kips (line 3) and $F = 92.8$ kips (line 5) as shown in Fig. 4(*d*).

(4) The design calculations corresponding to step 4 are shown in the first cycle of Table 2. The data for the design of girders, Table 2(*a*), include: M, the required S, $K\,L$, I_o, I, i, the section selected, and the actual S. For member AB, the required S is $\frac{822 \times 12}{20} = 493$, which is provided by a 36-in. WF at 150 lb, with $S = 503$ and $I = 9{,}012$. For member BC, the required S is $\frac{755 \times 12}{20} = 453$, which is likewise provided by a 36-in. WF at 150 lb. For member AB, $I_o = I = 9{,}012$; hence, $i = 0$. The inertia moment I_o for the remaining members is found by multiplying this value of I_o by the ratio of the value of $K\,L$ for the member in question to the value of $K_{AB}\,L_{AB}$. Thus, for member BC, $I_o = \frac{160}{180}\,(9{,}012) = 8{,}011$; and therefore $i = \frac{9{,}012 - 8{,}011}{8{,}011} = +\,0.125$.

The data for the design of columns, Table 2(*b*), include F, M, $K\,L$, A_c, A_b, the total of $A_c + A_b$, A'_c, A'_b, the total of $A'_c + A'_b$, I_o, I, i, L'/r, L'/b, the section selected, s_c, s_b, and the area provided. The columns are designed to resist, in addition to the axial force, (I) the moment at the top or bottom, or (II) 70% of the moment in case (I). In case (I), the required area is $A_c + A_b$. With allowable stresses of 17 kips per sq in. and 20 kips per sq in. for axial compression and bending, respectively, $A_c = \frac{F}{17}$ and $A_b = \frac{0.6\,M\,c}{r^2_b}$. In case

(II), the required area is $A'_c + A'_b$, where $A'_c = \frac{F}{s_c}$ and $A'_b = \frac{14}{s_b} A_b$. Case (I) was found to control in each instance.

TABLE 2.—Design Calculations Corresponding to Step (4)

Line	(*a*) Girders			(*b*) Columns					
1	Member	(I_o)	Section	Member	A_c	A'_c	(I_c)	L'/r	s_c
2	M	I	S	F	A_b	A'_b	I	L'/b	s_b
3	Required S	i		M	Total	Total	i	Section	Area
4	$K\,L$			$K\,L$					
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
				First Cycle					
1	AB:			AA′:					
2	822	(9,012)	....	136.5	8.0	8.9	(4,006)	58.5	15.3
3	493	9,012	36WF150	238	21.8	15.3	1,852	<15	20
4	180	0	503	80	29.8	24.2	−0.54	18WF105	30.9
1	BC:			BB′:					
2	755	(8,011)	....	297.8	17.5	19.5	(1,400)	59	15.3
3	453	9,012	36WF150	133	12.2	8.5	1,852	<15	20
4	160	+0.125	503	28	29.7	28.0	+0.323	18WF105	30.9
1				CC′:					
2	....	....	....	92.8	5.5	7.5	(1,001)	97	12.4
3	....	....	....	132	15.0	12.4	797	24	17.0
4	....	....	....	20	20.5	19.9	−0.204	14WF74	21.7
				Second Cycle					
1	AB:			AA′:					
2	867.1	(9,012)	....	136.5	8.0	8.9	(4,006)	59	15.3
3	520	9,739	36WF160	188.6	17.4	12.2	1,675	<15	20
4	180	+0.08	541	80	25.4	21.1	−0.044	18WF96	28.2
1	BC:			BB′:					
2	787.1	(8,011)	....	297.8	17.5	19.7	(1,400)	61	15.1
3	471	9,012	36WF150	165.0	15.1	10.6	2,034	<15	20
4	160	0	503	28	32.6	30.3	+0.13	18WF114	33.5
1				CC′:					
2	....	....	....	92.8	5.5	7.7	(1,001)	98	12.3
3	....	....	....	120.4	14.1	11.6	724	24	17.0
4	....	....	....	20	19.6	19.3	−0.073	14WF68	20.0
				Third Cycle					
1	AB:			AA′:					
2	869.3	(9,012)	....	136.5	8.0	8.9	(4,006)	59	15.3
3	521	9,739	36WF160	186.8	17.2	12.0	1,675	<15	20
4	180	0	541	80	25.2	20.9	0	18WF96	28.2
1	BC:			BB′:					
2	786.2	(8,011)	....	297.8	17.5	19.7	(1,400)	61	15.1
3	471	9,012	36WF150	179.4	16.4	11.5	2,034	<15	20
4	160	0	503	28	33.9	31.2	0	18WF114	33.5
1				CC′:					
2	....	....	....	92.8	5.5	7.7	(1,001)	98	12.3
3	....	....	....	115.9	13.5	11.1	724	24	17.0
4	....	....	....	20	19.0	18.8	0	14WF68	20.0

In the design of the columns some effort was made to conform with the relative stiffnesses assumed in Fig. 3(*a*), but the immediate objective was to obtain a design in which the area agreed closely with that required. With

reference to Table 2(*b*) (first cycle), for column AA′ the required area = 29.8 sq in.; $I_o = \frac{80}{180}(9{,}012) = 4{,}006$; the area provided (18-in. WF at 105 lb) = 30.9 sq in.; $I = 1{,}852$; and $i = \frac{1{,}852 - 4{,}006}{4{,}006} = -0.54$. For column BB′, the required area = 29.7 sq in.; $I_o = \frac{28}{180}(9{,}012) = 1{,}400$; the area provided (18-in. WF at 105 lb) = 30.9 sq in.; $I = 1{,}852$; and $i = \frac{1{,}852 - 1{,}400}{1{,}400} = +0.323$. For column CC′, the required area = 20.5 sq in.; $I_o = \frac{20}{180}(9{,}012) = 1{,}001$; the area provided (14-in. WF at 74 lb) = 21.7 sq in.; $I = 797$ and $i = \frac{797 - 1{,}001}{1{,}001} = -0.204$.

If subscripts 1 to 5 are used to refer to members AA′, AB, BB′, BC, and CC′, respectively, the stiffness-change ratios resulting from the first cycle of design are as follows: $i_1 = -0.54$; $i_2 = 0$; $i_3 = +0.323$; $i_4 = +0.125$; and $i_5 = -0.204$.

(5) Moment-stiffness relations are required: For M_{BA} and $M_{BB'}$, using values of M' and H' in Fig. 4(*a*); for M_{BC}, using values of M' and H' in Fig. 4(*b*); for $M_{AA'}$, using values of M' and H' in Fig. 4(*c*); and, for $M_{CC'}$, using values of M' and H' in Fig. 4(*d*). The calculation of the moment-stiffness relation for M_{BA} has already been explained in detail (see Example 3). The moment-stiffness relations for M_{BC}, $M_{AA'}$, $M_{BB'}$, and $M_{CC'}$ are calculated in a similar manner as shown in Table 1, together with the moment-stiffness relation for M_{BA}.

TABLE 3.—Moment Changes

Symbol	Original moment	$i_1 = -0.54$	$i_3 = +0.323$	$i_4 = +0.125$	$i_5 = -0.204$	Revised moment	$i_2 = +0.08$
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
M_{BA}	+822	+10.0	+12.1	+13.8	+ 9.2	+867.1	− 6.8
M_{BC}	−755	−27.4	+ 7.9	−17.2	− 0.3	−792.0	+ 4.9
$M_{AA'}$	+238	−38.4	+11.2	− 3.2	− 9.0	+198.6	−10.0
$M_{BB'}$	−133	+ 0.4	−28.2	+ 2.0	− 8.6	−167.4	+ 2.4
$M_{CC'}$	−132	− 9.9	+ 4.8	+ 2.4	+12.5	−122.2	+ 0.7

TABLE 3.—(*Continued*)

Symbol	Revised moment	$i_1 = -0.044$	Revised moment	$i_3 = +0.13$	Revised moment	$i_4 = -0.073$	Final moment
	(8)	(9)	(10)	(11)	(12)	(13)	(14)
M_{BA}	+860.3	+0.8	+861.1	+ 4.9	+866.0	+3.3	+869.3
M_{BC}	−787.1	−2.2	−789.3	+ 3.2	−786.1	−0.1	−786.2
$M_{AA'}$	+188.6	−3.1	+185.5	+ 4.5	+190.0	−3.2	+186.8
$M_{BB'}$	−165.0	0	−165.0	−11.3	−176.3	−3.1	−179.4
$M_{CC'}$	−121.5	−0.8	−122.3	+ 1.9	−120.4	+4.5	−115.9

(6) The calculated changes in the controlling moments are shown in Table 3. The original values of the moments are shown in Col. 1. Col. 2 contains the moment changes due to decreasing the stiffness of member AA′ by 54%

($i_1 = -0.54$). These moment changes were obtained by multiplying the coefficients of i_1, in Table 1, by -0.54. Cols. 3, 4, and 5, Table 3, which were calculated in a similar manner, show the remaining moment changes resulting from the first cycle of design, and Col. 6 shows the revised values of the moments. The values of F were assumed to remain essentially unchanged.

(7) In the second cycle of design, for which the calculations are shown in Table 2, the moments were revised after each change in section. For member AB, $M = 867.1$ ft-kips (Col. 6, Table 3), requiring a 36-in. WF at 160 lb, with $i_2 = +0.08$, which resulted in the moment changes and revised moments shown in Cols. 7 and 8, Table 3. For member BC, $M = 787.1$ ft-kips, and no change in design was required. For member AA′, $M = 188.6$ ft-kips, requiring an 18-in. WF at 96 lb, with $i_1 = -0.044$, which produced the moment changes and revised moments shown in Cols. 9 and 10, Table 3. For member BB′, $M = 165.0$ ft-kips, requiring an 18-in. WF at 114 lb, with $i_3 = +0.13$, and producing the moment changes and revised moments given in Cols. 11 and 12, Table 3. For member CC′, $M = 120.4$ ft-kips, requiring a 14-in. WF at 68 lb, with $i_5 = -0.073$, and producing the moment changes and revised moments given in Cols. 13 and 14 of Table 3.

Calculations for a third cycle of design, shown in Table 2, indicate that no additional changes were required.

For comparison with the values of F and M shown in the third cycle of Table 2, and in Col. 14, Table 3, the results of a direct final analysis were found to be as follows:

Member	F	M
AB	....	867
BC	....	796
AA′	133.7	184
BB′	302.7	176
CC′	93.6	113

Economy Studies

In the course of the foregoing design, alternate selections were available which, if adopted, would have resulted in a final design differing somewhat from that shown. Furthermore, it is to be expected that several designers, although adhering closely to the procedure herein described, but each starting with a different set of assumed stiffnesses, would obtain designs differing more or less from each other. In connection with an existing design there arises, therefore, the question of whether, by changing the stiffnesses of certain members, a saving can be effected in the total amount of main material. Such a saving may occur in two principal ways: First, a change in stiffness due to the substitution of a lighter member may be accompanied by a change in moment such that the resulting unit stresses in the lighter member may meet the requirements of the specifications, without producing overstress in the remaining members; and, second, it may be found possible to reduce the stresses in a given member (and thus to use a lighter member) by changing the stiffness of adjacent members, in addition to that of the member to be made lighter.

In the latter case, sometimes the weight per foot of a long member may be decreased with the aid of an increase in the weight per foot of an adjacent, shorter, member, with a resultant saving in total weight.

The writer's experience indicates that, in one or both of these ways, a saving of from approximately 3% to 6% of the total weight can frequently be obtained.

It does not appear practicable to formulate a fixed procedure for such investigations, except to state that it is essentially a matter of trial. Moment-stiffness relations are well suited to a rapid determination of the effects of various changes aimed at securing increased economy. In the foregoing example it is found, with the aid of the moment-stiffness relations in Table 1, that the member AA′ can be designed with a 24-in. WF section at 80 lb, with a saving of 640 lb, or about 4% of the total weight of the main members in the bent.

Conclusion

Experience indicates that moment-stiffness relations are useful aids in the design of continuous framed structures composed of prismatic members, particularly in cases where the columns are subject to moments of considerable magnitude. These relations also provide a basis for the determination of possible improvements in an existing design.

As differential quantities are only approximate representations of the true changes, the relative stiffnesses upon which the moment-stiffness relations are based should be chosen with some care, or determined from a preliminary design; and the final results should generally be checked by a direct analysis. It should be recognized that, in some instances, where large stiffness changes occur, the moment differentials will be subject to rather large errors; also, that, where only minor changes in stiffness occur, the use of moment-stiffness relations is not particularly advantageous unless subsequent economy studies are to be made.

APPENDIX 1

Notation

The following letter symbols, used in this paper and in its discussions, conform essentially to American Standard Letter Symbols for Mechanics, Structural Engineering and Testing Materials (ASA—Z10a—1932), prepared by a Committee of the American Standards Association, with Society representation, and approved by the Association in 1932.

A = section area required for given design conditions:

- A_b = area for bending at a section with lateral support;
- A_c = area for axial compression at a section with lateral support (see Table 2);
- A'_b = area for bending at an unsupported section;
- A'_c = area for axial compression at an unsupported section;

b = width of a section;

C = fixed-end moment caused by loads between the ends of a member;

c = distance from the gravity axis of a member to the outer fiber in bending;

E = modulus of elasticity;

F = axial force that controls the design of a member;

H = horizontal shear at the end of a vertical or inclined member;

H' = the difference between H and the fixed-end shear, Q (that is, $H' = H - Q$);

H'' = horizontal shear correction resulting from a distribution of shears;

h = column heights; the length of the vertical projection of an inclined member; h' = the vertical distance from the bottom of one inclined member to the prolongation of the other inclined member in the same span (see Fig. 6);

I = rectangular moment of inertia of a member as designed; I_o = the moment of inertia of a member corresponding to the stiffness assumed in the basic structure (that is, I_o is proportional to $K\,L$);

i = the stiffness-change ratio of a member = $\dfrac{dK}{K}$;

K = the relative flexural stiffness, or the $\dfrac{E\,I}{L}$-value of a member, as assumed in the basic structure;

L = length of a given member;

L' = unsupported length of a member;

dL = relative horizontal displacement of the opposite ends of a given span;

l = length of the horizontal projection of an inclined member;

M = moment; moment at the end of a member that controls its design;

M' = difference between M and C (that is, $M' = M - C$);

M'' = the fixed-end moments created by horizontal displacements of column tops, visualized in balancing horizontal shears;

δM = a change in moment resulting from the distribution of moments;

m = moment created by a unit couple applied at a place whose location is given by a subscript; that is, m_A is the moment caused by a unit couple at joint A and m_Δ is the moment created by a unit horizontal force that produces sidesway;

P = concentrated loads applied to various members of a structure;

Q = fixed-end horizontal shear due to loads between the ends of a vertical or inclined member, or the sum of the simple beam shear and the shear created by fixed-end moments; ΣQ is the sum of the fixed-end horizontal shears on a given joint or section and any external horizontal forces that may be acting on it;

r = least radius of gyration; r_b = radius of gyration in the plane of bending;

S = section modulus, I/c;

s = allowable unit stress at an unsupported section:

s_b = bending stress;

s_c = compressive stress;

V = vertical shear;

x = lever arm of a vertical load P from left support; x' = lever arm from right support (see Fig. 8);

y = vertical distance to a horizontal load P;

γ = side-thrust stiffness;

Δ = sidesway or displacement produced by a unit horizontal force; horizontal displacement of the top of a column;

κ = average rate of convergence;

ϕ = a linear function of the elastic deformations, given by the relation $H' = K\phi$, and

ψ = a linear function of the elastic deformations, given by the relation $M' = K\psi$.

APPENDIX 2

Analysis of Multiple-Span Peaked Bents

Multiple-span, peaked, or saw-tooth bents may be analyzed with a fair degree of accuracy and facility by alternate distributions of moments and horizontal shears. (The distribution of joint forces was suggested by Hardy Cross, M. ASCE, in 1926, in a mimeographed treatise entitled "Statically Indeterminate Structures.") The procedure involves the following steps:

1. Compute fixed-end moments at all joints and balance by moment distribution, assuming that the joints are not displaced.

2. Compute the horizontal shears at the tops of the columns and at the ends of the spans and balance them by distribution to the columns and adjoining spans in proportion to the side-thrust stiffness of these units; by addition, find the resulting shear corrections.

3. Compute the fixed-end moments due to the shear corrections and displacements occurring in the preceding distribution of horizontal shears and balance by moment distribution, as before.

4. From the changes in the moments resulting from the preceding distribution of moments, compute the parts of the horizontal shears remaining to be balanced and balance them by horizontal shear distribution, as before.

5. Repeat steps 3 and 4 until the solution converges. Horizontal displacements and horizontal end shears are considered positive to the right.

The side-thrust stiffness of a column is defined as the horizontal force or shear which, applied at the top of the column, will produce a unit horizontal displacement, without rotation, of the top of the column. The side-thrust stiffness of a span (consisting of two inclined members) is defined as the horizontal force or shear required to produce unit relative horizontal displacement

of the ends of the span without rotation of the ends or of the peak joint. In other words, if the ends of the span are denoted by the letters A and C, and the peak joint by the letter B, the side-thrust stiffness of the span is the value of of one of the two equal and opposite horizontal forces or shears which, applied at points A and C, will produce a unit change in the length of the span, without rotation of the joints A, B, and C, and without any vertical movement of joints A and C.

The similarities between moment distribution and horizontal shear distribution as applied to multiple-span peaked bents are apparent from the following brief description of the two processes:

Item	Moment distribution	Horizontal shear distribution
1.	The joints are balanced consecutively with regard to moments.	The joints at the tops of the columns are balanced consecutively with regard to horizontal shears.
2.	In balancing a given joint the only joint movement permitted is the rotation of the joint in question.	In balancing the joints at the tops of columns the only joint movements permitted are a horizontal displacement of the joint in question, and a vertical and horizontal displacement of the peak joints in adjacent spans. The joints are not permitted to rotate.
3.	The unbalanced moment is distributed to the connecting members in proportion to their flexural stiffness or resistance to rotation.	The unbalanced horizontal shear is distributed to the column and adjacent spans in proportion to their side-thrust stiffness, or resistance to horizontal displacement.
4.	The balancing of the moments produces fixed-end moments, or carry-over moments, at the opposite ends of the members; hence, whenever the joints are all balanced individually, a new set of unbalanced moments arises from the carry-over moments.	The balancing of the horizontal shears produces horizontal shears, or carry-over shears, at the opposite ends of the adjoining spans; hence, whenever the joints at the tops of the columns are all balanced individually, a new set of unbalanced horizontal shears arises from the carry-over shears. (A shear correction applied at one end of a span produces an equal and opposite shear at the other end of the span.)
5.	To obtain a complete distribution the process of balancing and carrying over must be continued until the corrections become negligible.	To obtain a complete distribution the process of balancing and carrying over must be continued until the corrections become negligible. This process may be expedited by the summation of a series.

Item	Moment distribution	Horizontal shear distribution
6.	Before the second, or any subsequent, distribution of moments can be made it is necessary to compute the fixed-end moments produced by the displacements occurring in connection with a corresponding distribution of horizontal shears.	Before a distribution of horizontal shears can be made it is necessary to compute the horizontal shears, or parts of them, which are required to be balanced.
7.	These fixed-end moments (item 6) are computed by simple formulas obtained from a consideration of the elastic deformations (see Eqs. 19, 23*a*, and 23*b*, derived subsequently).	These quantities (item 6) are obtainable by statics (see Eqs. 12, 14, 15*a*, and 15*b*, derived subsequently).

In the computation of items 1 to 7, numerical subscripts are used to designate the various members; for example, in Fig. 6, the column AA′ is designated by the subscript 1; and members AB and BC, by the subscripts 2 and 3, respectively.

The horizontal shears required in the first distribution of shears are obtained (by statics) from the loads on the structure and from the moments resulting from the first distribution of moments. The horizontal shear at the top of a column AA′ is as follows (see Figs. 6 and 7):

$$H_{AA'} = -\frac{M_{AA'} + M_{A'A} + \Sigma(P\,y)}{h_1} \quad \ldots\ldots(12)$$

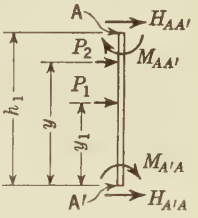


Fig. 7

Expressions for the horizontal shears at the lower ends of the inclined members are obtained by combining the equations of statics pertaining to the separate members and eliminating the unwanted horizontal and vertical forces. Thus, in Fig. 8 (involving vertical loads), by taking moments about points A and C:

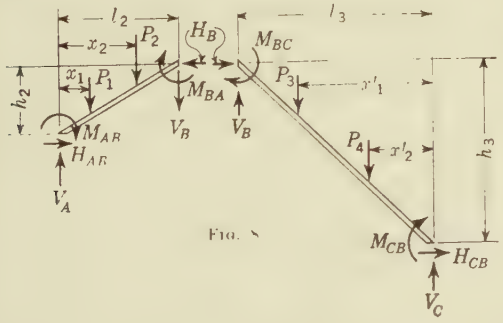


Fig. 8

$$M_{AB} + M_{BA} + V_B\,l_2 - H_B\,h_2 + \Sigma(P\,x) = 0 \ldots(13a)$$

and

$$M_{BC} + M_{CB} + V_B\,l_3 + H_B\,h_3 - \Sigma(P\,x') = 0 \ldots(13b)$$

Elimination of V_B and substitution from the relations $H_{AB} = -H_{CB} = H_B$ result in the following expression for the desired end shears:

$$H_{AB} = -H_{CB} = \frac{M_{AB} + M_{BA} + \Sigma(P\,x)}{h'_2} - \frac{M_{BC} + M_{CB} - \Sigma(P\,x')}{h'_3} \ldots(14)$$

in which $h'_2 = h_2 + \frac{l_2}{l_3} h_3$ and $h'_3 = h_3 + \frac{l_3}{l_2} h_2$ (see Fig. 6).

Equations for the calculation of the parts of the horizontal shears required to be balanced in subsequent distributions of horizontal shears are obtained in a similar manner; then, however, the quantities $\Sigma(P\,y)$, $\Sigma(P\,x)$, and $\Sigma(P\,x')$ are zero. Therefore, the horizontal shears which require balancing at the end of the second, or of a subsequent, distribution of moments are given by:

$$H_{AA'} = -\frac{\delta M_{AA'} + \delta M_{A'A}}{h_1} \qquad (15a)$$

and

$$H_{AB} = -H_{CB} = \frac{\delta M_{AB} + \delta M_{BA}}{h'_2} - \frac{\delta M_{BC} + \delta M_{CB}}{h'_3} \qquad (15b)$$

in which $\delta M_{AA'}$, $\delta M_{A'A}$, δM_{AB}, etc., are the changes in moment resulting from the distribution of moments.

Consider now the side-thrust stiffnesses and the fixed-end moments produced by the displacements occurring in the distribution of shears. The fixed-end moments of the column in Fig. 7, produced by the displacement Δ_A, are

$$M''_{AA'} = M''_{A'A} = -\frac{6\,K_1}{h_1}\,\Delta_A \qquad (16)$$

The corresponding shear correction, denoted by $H''_{AA'}$, is obtained by substituting these values of $M''_{AA'}$ and $M''_{A'A}$ (Eq. 16) in place of $\delta M_{AA'}$ and $\delta M_{A'A}$ in Eq. 15*a*:

$$H''_{AA'} = \frac{12\,K_1}{h^2_1}\,\Delta_A \qquad (17)$$

The side-thrust stiffness of column AA', found by placing $\Delta_A = 1$ in Eq. 17, is, therefore:

$$\gamma_{AA'} = \frac{12\,K_1}{h^2_1} \qquad (18)$$

The fixed-end moments accompanying the displacement Δ_A are obtained in terms of $H''_{AA'}$ by eliminating Δ_A from Eqs. 16 and 17, as follows:

$$M''_{AA'} = M''_{A'A} = -\frac{H''_{AA'}\,h_1}{2} \qquad (19)$$

By moment areas, or otherwise, it can be shown that, if the shear corrections, H''_{AB} and H''_{CB}, are applied at the ends of the span ABC, Fig. 6, in such a way as to produce a change in length, dL, of the span, L, without rotation of the joints A, B, C, and without any vertical movement of A and C, the following fixed-end moments will be obtained:

$$M''_{AB} = M''_{BA} = -\frac{6\,K_2}{h'_2}\,dL \qquad (20a)$$

and

$$M''_{BC} = M''_{CB} = +\frac{6\,K_3}{h'_3}\,dL \qquad (20b)$$

The corresponding shear corrections are obtained by substitution of these

values of M'' (Eqs. 20) in place of the values of δM in Eq. 15*b*:

$$H''_{AB} = -12\left[\frac{K_2}{(h'_2)^2} + \frac{K_3}{(h'_3)^2}\right] dL \quad \ldots\ldots\ldots\ldots (21a)$$

and

$$H''_{CB} = +12\left[\frac{K_2}{(h'_2)^2} + \frac{K_3}{(h'_3)^2}\right] dL \quad \ldots\ldots\ldots\ldots (21b)$$

The side-thrust stiffness of span ABC may be obtained by placing $dL = -1$ in Eq. 21*a* or by placing $dL = +1$ in Eq. 21*b*, giving

$$\gamma_{AC} = 12\left[\frac{K_2}{(h'_2)^2} + \frac{K_3}{(h'_3)^2}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (22)$$

The fixed-end moments accompanying the displacement dL are obtained in terms of the shear corrections, in the case of the member AB, by eliminating dL from Eqs. 20*a* and 21*a*, and, in the case of the member BC, by eliminating dL from Eqs. 20*b* and 21*b*, as follows:

$$M''_{AB} = M''_{BA} = \frac{H''_{BA}}{2} \frac{\dfrac{K_2}{h'_2}}{\dfrac{K_2}{(h'_2)^2} + \dfrac{K_3}{(h'_3)^2}} \quad \ldots\ldots\ldots\ldots (23a)$$

and

$$M''_{BC} = M''_{CB} = \frac{H''_{CB}}{2} \frac{\dfrac{K_3}{h'_3}}{\dfrac{K_2}{(h'_2)^2} + \dfrac{K_3}{(h'_3)^2}} \quad \ldots\ldots\ldots\ldots (23b)$$

Example 6.—Determine the bending moments at the ends of the members of the structure illustrated in Figs. 5(*a*) and 9(*a*), the loading consisting of 1 kip per ft of horizontal projection, uniformly distributed over both spans.

The relative flexural stiffnesses are shown in Fig. 9(*a*), and the relative side-thrust stiffnesses of the three columns and two spans are shown in Fig. 9(*b*). The fixed-end moments of members AB, BC, CD, and DE, due to the given loading, are distributed in Fig. 9(*a*).

The horizontal shears acting at the tops of the columns and at the ends of the spans at the beginning of the first shear distribution are shown in parentheses in Fig. 9(*b*). They are calculated in accordance with Eqs. 12 and 14. The unbalanced horizontal shears, or the algebraic sums of the quantities in parentheses, are written above the various joints in Fig. 9(*b*). The joints were balanced in the order A, E, and C. In the first cycle, the unbalanced shear of +29.54 kips at joint A is distributed to the column AA′ and the span ABC in proportion to their side-thrust stiffnesses—namely, 1.00 and 2.67—the corrections being −8.05 and −21.49, respectively. The unbalanced shear of −24.75 kips at joint E is distributed to the column EE′ and the span EDC in a like manner, the corrections being +6.74 and +18.01, respectively. Corrections assigned at one end of a given span are carried over to the other end with a change in sign; hence, at joint C, the unbalanced shear is + 0.28 + 21.49 − 18.01 = + 3.76 kips. This quantity is distributed to column CC′ and the

spans CBA and CDE in the proportions of 1.00, 2.67, and 2.67—the corrections being −0.59, −1.58, and −1.59, respectively. After the last two corrections are carried over to joints A and C, respectively, with change in sign, the second and third cycles of shear distribution are performed. The corrections

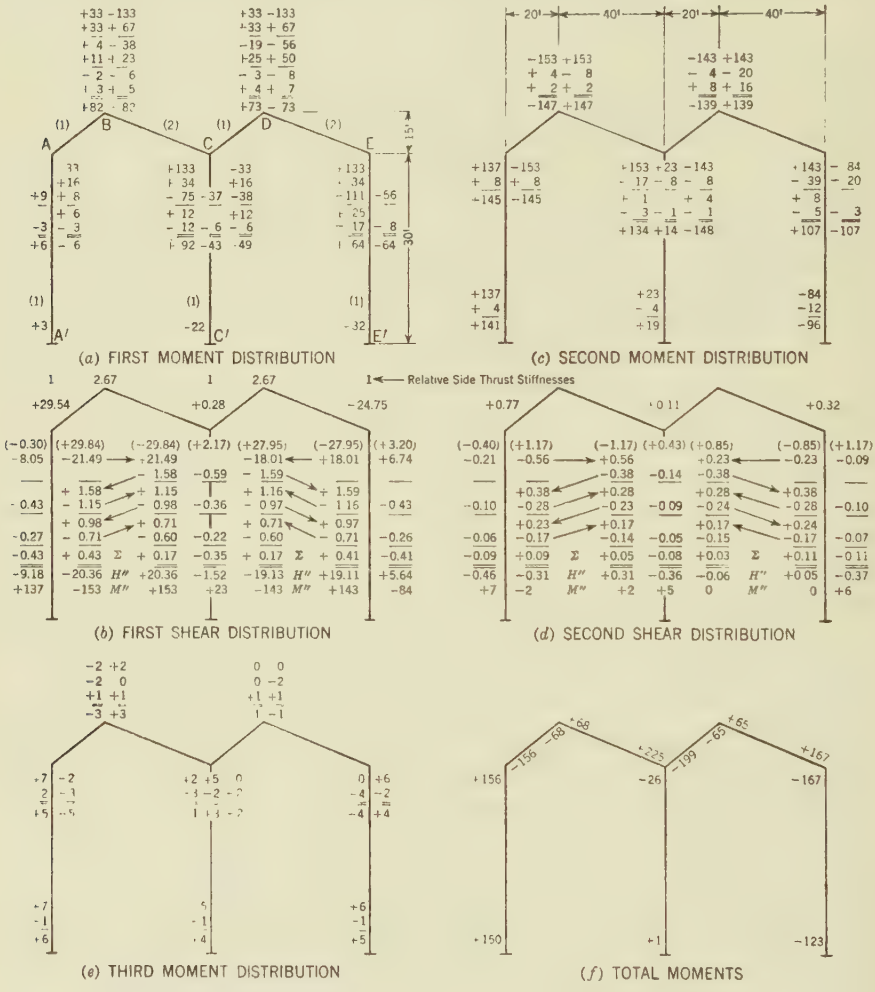


FIG. 9.—ANALYSIS OF MULTIPLE-SPAN PEAKED BENTS

now form geometric series, converging at approximately equal rates at the various joints. As the convergence is rather slow, a considerable saving in arithmetic may be made if the remainders are calculated.

The average rate of convergence, κ, may be found with sufficient precision by dividing the sum of the unbalanced shears (taken without regard to sign)

in the third cycle by the sum of the unbalanced shears in the second cycle. Hence, in the example, $\kappa = \frac{098 + 1.42 + 0.97}{1.58 + 2.31 + 1.59} = 0.613$. The remainders are summed by multiplying the results of the third cycle of distributions by $\frac{\kappa}{1 - \kappa} = \frac{0.613}{0.387} = 1.58$, and are designated by Σ in Fig. 9(*b*).

The total horizontal shear corrections are next found, by addition of each column of figures, exclusive of the quantities in parentheses; they are designated by H'' in Fig. 9(*b*). As a check on the shear distribution, the sum of values of H'' at a given joint should be equal and opposite to the original unbalanced shear. The sums in question are -29.54, -0.29, and $+24.75$. The fixed-end moments arising from the balancing of the horizontal shears are marked M'' in Fig. 9(*b*); they are calculated in accordance with Eqs. 19, 23*a*, and 23*b*.

The M''-values obtained in Fig. 9(*b*) are written as fixed-end moments in Fig. 9(*c*), and the joints are again balanced by moment distribution. The unbalanced horizontal shears are then calculated from the changes in the moments of Fig. 9(*c*), in accordance with Eqs. 15*a* and 15*b*, and are written in parentheses in Fig. 9(*d*). The calculations are as follows:

$$H_{AA'} = \frac{-12}{30} = -0.40 \text{ kip}; \quad H_{CC'} = \frac{+13}{30} = +0.43 \text{ kip};$$

$$H_{EE'} = \frac{+35}{30} = +1.17 \text{ kips};$$

$$H_{AB} = \frac{8 + 6}{22.5} - \frac{-6 - 19}{45} = +1.17 \text{ kips}; \quad H_{CB} = -1.17 \text{ kips};$$

$$H_{CD} = \frac{-5 + 4}{22.5} - \frac{-4 - 36}{45} = +0.85 \text{ kip}; \quad H_{ED} = -0.85 \text{ kip}.$$

The second shear distribution is shown in Fig. 9(*d*); the computations were made in the same way as those for the first shear distribution. The resulting fixed-end moments, marked M'' in Fig. 9(*d*), are balanced by moment distribution in Fig. 9(*e*). The solution is summarized in Fig. 9(*f*), the moments shown being the sums of the quantities appearing in Figs. 9(*a*), 9(*c*), and 9(*e*).

VITA

George Harper Dell was born in Mapleton, Pa., on May 10, 1894. Following graduation from the Mapleton High School in 1911 he was employed in clerical work and in the Federal civil service until he matriculated, in 1916, at The Pennsylvania State College. From 1917 to 1919 he was enlisted in the U. S. Army Ambulance Service. In 1919 he resumed his studies in civil engineering and in 1922 was graduated with highest honors from The Pennsylvania State College, with the degree of Bachelor of Science.

During the summers of 1920, 1921, and 1922 he was employed on highway projects in Pennsylvania. From August, 1922, to April, 1925, he was an officer in the U. S. Coast and Geodetic Survey, and from April, 1925, to September, 1927, was employed as assistant city engineer in Chattanooga, Tenn. From September, 1927, to date he has been a member of the faculty of the Department of Civil Engineering, University of Illinois, teaching surveying, structural design, and navigation.

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